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# AN INTRINSIC FORMULATION FOR NONLINEAR ELASTIC RODS

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**Abstract**—Using the theory of a Cosserat rod with two directors, special constrained Bernoulli–Euler type theory is developed for large spatial deformations which omits the effects of normal cross-sectional extension, tangential shear deformation, and the normal cross-sectional shear deformation but allows the nonlinear elastic strain energy to be a general function of the extension, curvature and twist of the rod. The resulting equilibrium equations are written in an intrinsic form in terms of the extension, curvature, twist, and the geometric torsion of the rod's reference curve. It is known that exact integrals of the equilibrium equations exist for the simple case when the rod is loaded only by forces and moments applied to its ends. Here, similar exact integrals yield implicit algebraic functions of curvature for the extension, twist and geometric torsion. The remaining equilibrium equation becomes a second order equation for curvature alone which can be analyzed completely. An example is presented which shows the influence of extensional deformation on axial force and bending moment. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

Love (1944) discusses the early works of Kirchhoff and Euler who analyzed large deformations of elastic rods in equilibrium. This formulation assumes that the reference curve of the rod is inextensible and that the cross-section of the rod remains perpendicular to the tangent to this reference curve so the effect of tangential shear deformation is absent. Within this context, the cross-section of the rod and the tangent vector to the rod's reference curve are characterized by an orthonormal triad of vectors that are specified by Euler angles.

In general, the force at an end of the rod separates into a tangential force which acts along the tangent to the rod's reference curve and a shear force which acts in the rod's normal cross-section. Also, the moment at an end of the rod separates into a torsional moment applied in the tangential direction and a bending moment applied in the normal cross-section. The combination of force and moment can cause the rod's reference curve to develop nonzero curvature and geometric torsion as defined by the associated Serret–Frenet triad. Consequently, it is necessary to distinguish between the twist about the tangential direction caused by the torsional moment and the geometric torsion of the rod's reference curve. With this background in mind, it is noted that the equations recorded in Love (1944) require the bending moment to be a linear function of a curvature and the torsional moment to be a linear function of the twist.

Antman (1972, 1974) has generalized this analysis of equilibrium of rods by including the effects of extension of the rod's reference curve and tangential shear deformation. Also, he has considered a rather general class of constitutive equations which depend nonlinearly on various kinematical variables. His analysis (Antman, 1974) uses the Euler angle formulation and focuses attention on the problem of a rod that is loaded only by forces and moments applied to its ends. A number of general results have been developed and restrictions on the constitutive functions relating to existence and uniqueness of solutions have been discussed. Moreover, his recent book (Antman, 1995) provides a comprehensive review of the analysis of rods and additional historical information can be found in the article by Dill (1992).

Of particular relevance to the current paper is the fact that Ericksen (1970) has developed an energy-type integral for general homogeneous elastic rods. Also, it is recalled (Antman, 1972; Whitman and DeSilva, 1974; Antman and Jordan, 1975) that other exact integrals of the equations of equilibrium can be obtained for the simple case of a rod loaded only on its ends.

The bending moment for an initially straight rod with general cross-sectional area is not necessarily oriented normal to the local plane of bending. However, if the cross-section of the rod is sufficiently symmetric [i.e., the principal values of the second moment of area of the cross-section (about its centroid) are equal] then the bending moment is oriented in the direction of the binormal to the reference space curve. Consequently, for such rods, it is quite natural to formulate the equations in an intrinsic form using the Serret–Frenet triad.

Recently, Lu and Perkins (1994) have analyzed the stability of equilibrium solutions of the rod equations with the aim of studying conditions that may eliminate undesirable hockling of fiber optic cables during ocean deployment. They seem to be the first authors to have reformulated the equations in Love (1944) in an intrinsic form. This intrinsic form presents differential equations that depend on the tangential force, the torsional moment, and the curvature and geometric torsion of the rod's reference curve. This formulation of homogeneous rods yields four differential equations of equilibrium: three first order equations and one second order equation. For the case of loading only on the ends of the rod, these first order equations integrate exactly and the results can be substituted into the fourth equation to obtain closed-form elliptical solutions which are consistent with "Kirchhoff's kinetic analogue" (Love, 1944).

The objective of the present paper is to generalize the formulation of Lu and Perkins (1994) to include arbitrary dependence of the nonlinear elastic strain energy function on the tangential extension, curvature and twist of the rod. This will be accomplished by specializing the theory of a Cosserat curve developed by Green *et al.* (1974a, b). The Bernoulli–Euler constraints, which eliminate normal cross-sectional extension, tangential shear deformation, and normal cross-sectional shear deformation (Naghdi and Rubin, 1984), will be employed but the dependence on tangential extension, curvature and twist will remain general. As to be expected, exact integrals of the associated intrinsic equilibrium equations exist when the rod is loaded only on its ends. In particular, it will be shown that two of the equilibrium equations integrate exactly for general inhomogeneous rods. If the rod is further assumed to be homogeneous then an energy-type integral also exists (Ericksen, 1970).

Thus, for homogeneous rods it follows that, in principle, the tangential extension, twist and geometric torsion can be written as implicit functions of the curvature so that the fourth equilibrium equation can be written as a second order equation for curvature alone. Consequently, the main features of the equilibrium solutions can be analyzed completely using the intrinsic formulation without having to solve the additional differential equations that determine the rod's reference curve in terms of the extension, curvature, and geometric torsion. Of course, when the boundary conditions require specific restrictions on the absolute location of the ends of the rod or the absolute directions of the forces and moments applied to the ends, then the equations for the rod's reference curve must be solved together with the intrinsic equations of equilibrium.

An outline of the paper is as follows: Section 2 presents the basic equations of a Cosserat curve, and Section 3 develops the intrinsic formulation of a generalized Bernoulli–Euler rod. Section 4 presents an alternative formulation that simplifies the constitutive equations, and Section 5 develops intrinsic forms of exact integrals of the equilibrium equations. Section 6 considers uniform solutions for circular helical rods and Section 7 considers the inextensible case. Section 8 presents a summary of the theory and uses special constitutive assumptions to discuss some influences of including extensional deformation and nonlinear dependence of the bending moment on the curvature. Appendix A presents some details of the calculations, and Appendix B briefly reviews Ericksen's energy integral within the context of the present Cosserat theory.

Throughout the text, bold faced symbols are used to denote tensors and the summation convention is implied for repeated indices. Latin indices take the values (1, 2, 3) and Greek indices take the values (1, 2). Also, the notation $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar product between two vectors **a** and **b**.

## 2. BASIC EQUATIONS

By way of background, it is recalled that a rod-like body (or rod) is a three-dimensional body that has special geometrical features. In particular, a rod can be considered to be a space curve with some cross-sectional area. Within the context of the theory of a Cosserat curve (Green *et al.*, 1974a, b) a rod-like body is *modeled* as a curve $c$ in space with any finite number of directors vectors that are used to include limited information about deformations through the thickness of the rod. Green *et al.* (1974a,b) have developed the theory of a Cosserat curve both by direct approach and by derivation from the three-dimensional theory and Naghdi (1982) has more recently reformulated the equations in direct notation.

For the simplest theory, the rod in its present configuration is modeled as a curve $c$ with material points located (relative to a fixed point) by the position vector $\mathbf{r}$ and two directors $\mathbf{d}_\alpha$, which are functions of the single convected coordinate $\xi$ and time $t$

$$\mathbf{r} = \hat{\mathbf{r}}(\xi, t), \quad \mathbf{d}_\alpha = \hat{\mathbf{d}}_\alpha(\xi, t), \tag{1a,b}$$

with the tangent vector $\mathbf{d}_3$ to the curve (1a) given by

$$\mathbf{d}_3 = \frac{\partial \hat{\mathbf{r}}}{\partial \xi}. \tag{2}$$

Moreover, it is assumed that the vectors $\mathbf{d}_i$ ($i = 1, 2, 3$) are linearly independent and form a right-handed triad such that

$$\mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{d}_3 > 0. \tag{3}$$

Similarly, the absolute velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ are defined by the expressions

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \hat{\mathbf{r}}}{\partial t}, \quad \mathbf{w}_i = \dot{\mathbf{d}}_i = \frac{\partial \hat{\mathbf{d}}_i}{\partial t}, \tag{4a,b}$$

where a superposed dot denotes material time differentiation holding $\xi$ fixed. Also, it is convenient to introduce the reciprocal vectors $\mathbf{d}^i$ defined by

$$\mathbf{d}_i \cdot \mathbf{d}^j = \delta_i^j, \tag{5}$$

where $\delta_i^j$ is the Kronecker delta.

Now, the local forms of the conservation of mass and the balances of linear momentum, director momentum and angular momentum, and an expression for the mechanical power $P$ can be written as

$$\lambda = \lambda(\xi), \tag{6a}$$

$$\lambda \hat{\mathbf{f}} + \frac{\partial \mathbf{k}^3}{\partial \xi} = 0, \quad \lambda \hat{\mathbf{f}} = \lambda \mathbf{f} - \lambda(\dot{\mathbf{v}} + y^\alpha \dot{\mathbf{w}}_\alpha), \tag{6b,c}$$

$$\lambda \hat{\mathbf{l}}^\alpha - \mathbf{k}^\alpha + \frac{\partial \mathbf{m}^\alpha}{\partial \xi} = 0, \quad \lambda \hat{\mathbf{l}}^\alpha = \lambda \mathbf{l}^\alpha - \lambda(y^\alpha \dot{\mathbf{v}} + y^{\alpha\beta} \dot{\mathbf{w}}_\beta), \tag{6d,e}$$

$$\mathbf{d}_i \times \mathbf{k}^i + \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \mathbf{m}^\alpha = 0, \tag{6f}$$

$$P = \mathbf{n}^i \cdot \mathbf{w}_i + \mathbf{m}^\alpha \cdot \frac{\partial \mathbf{w}_\alpha}{\partial \xi}, \tag{6g}$$

where $\lambda$ is the mass density per unit reference length of the curve $c$, $\mathbf{f}$ is the assigned force per unit mass due to body force and due to contact forces on the lateral surface, $\mathbf{k}^3$ is the contact force, $\mathbf{l}^\alpha$ are the assigned director couples per unit mass due to body force and due to contact forces on the lateral surface, $\mathbf{k}^\alpha$ are the intrinsic director couples per unit reference length of the curve $c$, $\mathbf{m}^\alpha$ are the contact couples, and $y^\alpha$ and $y^{\alpha\beta}$ are inertia coefficients that are functions of $\xi$ but are independent of time, with $y^{\alpha\beta}$ being a symmetric tensor. The contact force $\mathbf{k}^3$ and couples $\mathbf{m}^\alpha$ act on the end of the rod whose outward normal to the cross-section makes an acute angle with the tangential vector $\mathbf{d}_3$. This notation differs slightly from previous notations and is convenient for a theory with only two directors because it simplifies the expressions for angular momentum and the mechanical power by taking

$$\mathbf{k}^3 = \mathbf{n}, \tag{7}$$

where $\mathbf{n}$ is defined in (Naghdi, 1982).

For an elastic rod it is assumed that a strain energy $\psi$ per unit mass exists of the form

$$\psi = \bar{\psi}(\mathbf{d}_i, \partial\mathbf{d}_\alpha/\partial\xi, \xi), \tag{8}$$

where explicit dependence on the reference geometry of the rod has been included through the variable $\xi$. Furthermore, it is recalled that under superposed rigid body motions the directors $\mathbf{d}_i$ transform $\mathbf{d}_i^+$ such that

$$\mathbf{d}_i^+ = \mathbf{Q}(t)\mathbf{d}_i, \tag{9}$$

where $\mathbf{Q}$ is a proper orthogonal function of time only. Thus, it can be shown that in order for $\psi$ to remain form-invariant under superposed rigid body motions it must depend on the vectors $\mathbf{d}_i$ and $\partial\mathbf{d}_\alpha/\partial\xi$ only through the scalars

$$d_{ij} = \mathbf{d}_i \cdot \mathbf{d}_j, \quad \lambda_{\alpha j} = \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \cdot \mathbf{d}_j, \tag{10a,b}$$

so that $\psi$ can be replaced by the alternative function

$$\psi = \hat{\psi}(d_{ij}, \lambda_{\alpha j}, \xi). \tag{11}$$

Now for an elastic rod it is assumed that the mechanical power $P$ is balanced by the rate of change of the strain energy function $\psi$ such that

$$P = \lambda\dot{\psi}, \tag{12}$$

and that the vectors $\mathbf{k}^i$ and $\mathbf{m}^\alpha$ are independent of the rates $\mathbf{w}_i$ and $\partial\mathbf{w}_i/\partial\xi$.

For constrained materials it is further assumed that the quantities $\mathbf{k}^i$, and $\mathbf{m}^\alpha$ separate additively into two parts; one part $\hat{\mathbf{k}}^i$ and $\hat{\mathbf{m}}^\alpha$ which are determined by constitutive equations; and the other part $\bar{\mathbf{k}}^i$ and $\bar{\mathbf{m}}^\alpha$ which are constraint responses that are workless and satisfy the balance of the angular momentum so that

$$\mathbf{k}^i = \hat{\mathbf{k}}^i + \bar{\mathbf{k}}^i, \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha, \tag{13a,b}$$

$$\bar{\mathbf{k}}^i \cdot \mathbf{w}_i + \bar{\mathbf{m}}^\alpha \cdot \frac{\partial \mathbf{w}_\alpha}{\partial \xi} = 0, \quad \mathbf{d}_i \times \bar{\mathbf{k}}^i + \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \bar{\mathbf{m}}^\alpha = 0. \tag{13c,d}$$

Next, substitution of (11) into (12) and use of (13) yields the expression

$$\hat{\mathbf{k}}\cdot\mathbf{w}_i+\hat{\mathbf{m}}^{\alpha}\cdot\frac{\partial\mathbf{w}_\alpha}{\partial\xi}=\lambda\frac{\partial\bar{\psi}}{\partial\mathbf{d}_i}\cdot\mathbf{w}_i+\frac{\partial\bar{\psi}}{\partial(\partial\mathbf{d}_\alpha/\partial\xi)}\cdot\frac{\partial\mathbf{w}_\alpha}{\partial\xi}$$
$$=\lambda\left[2\frac{\partial\hat{\psi}}{\partial d_{ij}}\mathbf{d}_j+\frac{\partial\hat{\psi}}{\partial\lambda_{\alpha i}}\frac{\partial\mathbf{d}_\alpha}{\partial\xi}\right]\cdot\mathbf{w}_i+\lambda\left[\frac{\partial\hat{\psi}}{\partial\lambda_{\alpha j}}\mathbf{d}_j\right]\cdot\frac{\partial\mathbf{w}_\alpha}{\partial\xi}. \quad (14)$$

Then, using standard arguments it follows that

$$\hat{\mathbf{k}}^i=\lambda\frac{\partial\bar{\psi}}{\partial\mathbf{d}_i}=\lambda\left[2\frac{\partial\hat{\psi}}{\partial d_{ij}}\mathbf{d}_j+\frac{\partial\hat{\psi}}{\partial\lambda_{\alpha i}}\frac{\partial\mathbf{d}_\alpha}{\partial\xi}\right], \quad (15a)$$

$$\hat{\mathbf{m}}^{\alpha}=\lambda\frac{\partial\bar{\psi}}{\partial(\partial\mathbf{d}_i/\partial\xi)}=\lambda\left[\frac{\partial\hat{\psi}}{\partial\lambda_{\alpha j}}\mathbf{d}_j\right]. \quad (15b)$$

Moreover, since the functional form $\hat{\psi}$ in (11) is properly invariant under superposed rigid body motions it follows that the constitutive expressions (15) automatically satisfy the balance of angular momentum

$$\mathbf{d}_i\times\hat{\mathbf{k}}^i+\frac{\partial\mathbf{d}_\alpha}{\partial\xi}\times\hat{\mathbf{m}}^{\alpha}=0, \quad (16)$$

where use has been made of the fact that $\partial\hat{\psi}/\partial d_{ij}$ is a symmetric tensor.

For equilibrium problems the vectors $\mathbf{r}$ and $\mathbf{d}_\alpha$ are independent of time and the balances of linear momentum (6b) and director momentum (6d) reduce to

$$\lambda\mathbf{f}+\frac{\mathrm{d}\mathbf{k}^3}{\mathrm{d}\xi}=0,\quad \lambda\mathbf{l}^{\alpha}-\mathbf{k}^{\alpha}+\frac{\mathrm{d}\mathbf{m}^{\alpha}}{\mathrm{d}\xi}=0. \quad (17a,b)$$

## 3. AN INTRINSIC FORMULATION OF A GENERALIZED BERNOULLI-EULER ROD

The objective of this section is to develop an intrinsic form of the equilibrium equations for a generalized Bernoulli-Euler rod. In particular, the nonlinear elastic strain energy is allowed to be a general function of the stretch, the curvature of the reference curve, and the twist of the rod. Then, the equilibrium equations are written in an intrinsic form in terms of these kinematic quantities as well as in terms of the geometric torsion of the space curve $\mathbf{r}(\xi)$.

For the Bernoulli-Euler rod under consideration it is necessary to introduce the following constraints

$$d_{11}=1,\quad d_{22}=1, \quad (18a,b)$$

$$d_{13}=0,\quad d_{23}=0, \quad (18c,d)$$

$$d_{12}=0, \quad (18e)$$

that eliminate normal cross-sectional extension (18a, b), tangential shear deformation (18c, d), and normal cross-sectional shear deformation (18e) (see Naghdi and Rubin, 1984). Using (10a) and taking the material derivative of (18) it can be shown that these constraints can be written in the forms

$$\mathbf{d}_i\cdot\mathbf{w}_\alpha+\mathbf{w}_i\cdot\mathbf{d}_\alpha=0. \quad (19)$$

Then, the constraint responses associated with (19) become

$$\bar{\mathbf{k}}^i = p^{ij}\mathbf{d}_j, \quad \bar{\mathbf{m}}^\alpha = 0, \tag{20}$$

where $p^{ij}$ are Lagrange multipliers that are arbitrary functions of $\xi$ with $p^{33}$ vanishing and $p^{\alpha j}$ being symmetric in its indices

$$p^{33} = 0, \quad p^{\alpha j} = p^{j\alpha}. \tag{21a,b}$$

It then follows that the constraint responses (20) automatically satisfy the balance of angular momentum (13d). Furthermore, it can be shown by differentiating (18) with respect to $\xi$ and by using the definition (10b) that the constraints also require

$$\lambda_{11} = 0, \quad \lambda_{22} = 0, \tag{22a,b}$$

$$\frac{\mathrm{d}\mathbf{d}_3}{\mathrm{d}\xi}\cdot\mathbf{d}_\alpha = -\lambda_{\alpha 3}, \quad \lambda_{21} = -\lambda_{12}, \tag{22c,d}$$

so the constraint responses (20) are workless in the sense of (B2) since

$$\bar{\mathbf{k}}^i\cdot\frac{\mathrm{d}\mathbf{d}_i}{\mathrm{d}\xi} + \bar{\mathbf{m}}^\alpha\cdot\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\mathrm{d}\mathbf{d}_\alpha}{\mathrm{d}\xi}\right] = p^{ij}\frac{\mathrm{d}\mathbf{d}_j}{\mathrm{d}\xi}\cdot\mathbf{d}_j = p^{\beta\alpha}\lambda_{\beta\alpha} + p^{3\alpha}\left[\frac{\mathrm{d}\mathbf{d}_3}{\mathrm{d}\xi}\cdot\mathbf{d}_\alpha + \lambda_{\alpha 3}\right] = 0. \tag{23}$$

The values of $p^{\alpha j}$ are determined by satisfying the balances of director momentum (17b). In particular, with the help of (13) and (20), the equation (17b) can be rewritten in the form

$$\bar{\mathbf{k}}^\alpha = p^{\alpha j}\mathbf{d}_j = \lambda\mathbf{l}^\alpha - \hat{\mathbf{k}}^\alpha + \frac{\mathrm{d}\mathbf{m}^\alpha}{\mathrm{d}\xi}, \tag{24}$$

which can be solved using the reciprocal vectors $\mathbf{d}^j$ defined in (5) to deduce that

$$p^{\alpha j} = p^{j\alpha} = \mathbf{d}^j\cdot\left[\lambda\mathbf{l}^\alpha - \hat{\mathbf{k}}^\alpha + \frac{\mathrm{d}\mathbf{m}^\alpha}{\mathrm{d}\xi}\right]. \tag{25}$$

However, since $p^{\alpha j}$ must be symmetric it has only five independent components. Consequently, the six equations (25) yield five equations for $p^{\alpha j}$ and the sixth equation

$$p^{12} = p^{21} \Rightarrow \mathbf{d}^1\cdot\left[\lambda\mathbf{l}^2 - \hat{\mathbf{k}}^2 + \frac{\mathrm{d}\mathbf{m}^2}{\mathrm{d}\xi}\right] = \mathbf{d}^2\cdot\left[\lambda\mathbf{l}^1 - \hat{\mathbf{k}}^1 + \frac{\mathrm{d}\mathbf{m}^1}{\mathrm{d}\xi}\right], \tag{26a,b}$$

becomes an equation to determine the torsional moment (or twist) about the tangential axis of the rod. Therefore, for this constrained theory the equations of equilibrium reduce to (17a) and (26b).

The usual kinematic assumption can be used to express the three-dimensional position vector $\mathbf{r}^*$ in terms of the vectors $\mathbf{r}$, $\mathbf{d}_\alpha$ and the convected coordinates $\theta^\alpha$ in the cross-section and $\xi$ in the form

$$\mathbf{r}^* = \mathbf{r} + \theta^\alpha\mathbf{d}_\alpha. \tag{27}$$

Then, the definitions of the director couples $\mathbf{m}^\alpha$ in terms of the three-dimensional traction vector on the cross-section can be used to express the resultant moment $\mathbf{m}$ applied to the cross-section about the intersection of the cross-section with the curve $\mathbf{r}$ by

$$\mathbf{m} = \mathbf{d}_\alpha \times \mathbf{m}^\alpha, \quad \mathbf{m}^\alpha = \lambda \left[ \frac{\partial \hat{\psi}}{\partial \lambda_{\alpha j}} \right] \mathbf{d}_j. \tag{28}$$

Next, the boundary conditions are specified in terms of kinematics at the ends of the beam or in terms of the resultant force $\mathbf{n}$ and the resultant moment $\mathbf{m}$ with

$$\mathbf{n} = \mathbf{k}^3 = \lambda \left[ 2 \frac{\partial \hat{\psi}}{\partial d_{3j}} \mathbf{d}_j + \frac{\partial \hat{\psi}}{\partial \lambda_{\alpha 3}} \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \right] + p^{3\alpha} \mathbf{d}_\alpha. \tag{29}$$

The expressions (25), (28), (29) and the equations of equilibrium (17a) and (26b) are valid for a general functional form of the strain energy $\hat{\psi}$. In order to develop intrinsic equations which depend only on the stretch, the curvature and geometric torsion of the space curve $\mathbf{r}$, and the twist of the rod about the tangential axis, it is necessary to specialize this constitutive function. By way of background, it is recalled that the unit tangent vector $\mathbf{e}_t$, the unit normal vector $\mathbf{e}_n$ and the unit binormal vector $\mathbf{e}_\tau$ associated with the Serret–Frenet triad are defined such that

$$\mathbf{e}_t = \frac{\mathbf{d}_3}{d}, \quad \frac{\mathrm{d}\mathbf{e}_t}{\mathrm{d}s} = \frac{1}{d} \frac{\mathrm{d}\mathbf{e}_t}{\mathrm{d}\xi} = \kappa \mathbf{e}_n, \tag{30a,b}$$

$$\mathbf{e}_\tau = \mathbf{e}_t \times \mathbf{e}_n, \quad \frac{\mathrm{d}\mathbf{e}_n}{\mathrm{d}s} = \frac{1}{d} \frac{\mathrm{d}\mathbf{e}_n}{\mathrm{d}\xi} = -\kappa \mathbf{e}_t + \tau \mathbf{e}_\tau, \tag{30c,d}$$

$$\frac{\mathrm{d}\mathbf{e}_\tau}{\mathrm{d}s} = \frac{1}{d} \frac{\mathrm{d}\mathbf{e}_\tau}{\mathrm{d}\xi} = -\tau \mathbf{e}_n, \tag{30e}$$

where $\mathrm{d}s$ is the arclength in the present configuration and the stretch $d$ is defined by

$$d = (\mathbf{d}_3 \cdot \mathbf{d}_3)^{1/2} = \frac{\mathrm{d}s}{\mathrm{d}\xi}. \tag{31}$$

In order to motivate the desired form for the constitutive equations it is convenient to first develop an expression for the derivative of $\mathbf{d}_3$ in terms of the intrinsic variables and the vector triad $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_\tau\}$. To this end, it is noted from (31) that

$$\frac{\mathrm{d}\mathbf{d}_3}{\mathrm{d}\xi} \cdot \mathbf{d}_3 = d \frac{\mathrm{d}(\mathbf{d})}{\mathrm{d}\xi}, \tag{32}$$

so that with the help of (22c) it can be shown that

$$\frac{\mathrm{d}\mathbf{d}_3}{\mathrm{d}\xi} = -\lambda_{\alpha 3} \mathbf{d}^\alpha + d \frac{\mathrm{d}(d)}{\mathrm{d}\xi} \mathbf{d}^3. \tag{33}$$

However, in view of the constraints (18) and the definition (31) the reciprocal vectors $\mathbf{d}^i$ become

$$\mathbf{d}^\alpha = \mathbf{d}_\alpha, \quad \mathbf{d}^3 = \frac{1}{d^2} \mathbf{d}_3 = \frac{1}{d} \mathbf{e}_t. \tag{34a,b}$$

Thus,

$$\frac{\mathrm{d}\mathbf{d}_3}{\mathrm{d}\xi} = -\lambda_{\alpha 3}\mathbf{d}_\alpha + \frac{\mathrm{d}(d)}{\mathrm{d}\xi}\mathbf{e}_t. \tag{35}$$

Next, using (30a, b) it follows that

$$\kappa\mathbf{e}_n = \frac{1}{d^2}\frac{\mathrm{d}\mathbf{d}_3}{\mathrm{d}\xi} - \frac{1}{d^2}\frac{\mathrm{d}(d)}{\mathrm{d}\xi}\mathbf{e}_t = -\frac{\lambda_{\alpha 3}}{d^2}\mathbf{d}_\alpha. \tag{36}$$

Consequently, the curvature $\kappa$ can be expressed as a function of the form

$$\kappa^2 = \frac{\lambda_{\alpha 3}\lambda_{\alpha 3}}{d^4}. \tag{37}$$

Also, it follows from the definition (10b) and the result (22d), that it is convenient to define the kinematic quantity

$$\omega = \frac{k}{2}(\lambda_{12} - \lambda_{21}), \tag{38}$$

where $k$ is a constant having the units of length (e.g., $k$ can be the radius of gyration of a uniform rod's cross-sectional area; $I = k^2 A$ where $I$ is the second moment of the cross-sectional area $A$). Thus, $\omega$ is a nondimensional measure of the twist of the rod due to a torsional moment about the tangential axes $\mathbf{e}_t$.

These results are used to motivate consideration of a special strain energy function that depends only on the stretch $d$, the curvature $\kappa$, the twist $\omega$, and $\xi$. Thus, as a special case it is assumed that

$$\lambda\psi = \Sigma(d, \beta, \omega, \xi) = \hat{\Sigma}(d_{33}, \lambda_{12}, \lambda_{21}, \lambda_{\alpha 3}, \xi), \tag{39}$$

where for convenience $\beta$ is defined to be a nondimensional measure of the curvature

$$\beta = k\kappa. \tag{40}$$

The constitutive form (39) tacitly assumes that the cross-section of the rod is symmetrical in the sense that the response to bending in any plane containing the normal to the cross-section is the same.

Next, substitution of (39) into (15a), (29) and (28b) yields the expressions

$$\hat{\mathbf{k}}^1 = -\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\frac{\mathrm{d}\mathbf{d}_2}{\mathrm{d}\xi}, \quad \hat{\mathbf{k}}^2 = \frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\frac{\mathrm{d}\mathbf{d}_1}{\mathrm{d}\xi}, \tag{41a,b}$$

$$\mathbf{n} = \mathbf{k}^3 = \left[\frac{\partial\Sigma}{\partial d} - \frac{2\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_t + \left[\frac{k^2\lambda_{\alpha 3}}{\beta d^4}\frac{\partial\Sigma}{\partial\beta}\right]\frac{\mathrm{d}\mathbf{d}_\alpha}{\mathrm{d}\xi} + p^{3\alpha}\mathbf{d}_\alpha, \tag{41c}$$

$$\mathbf{m}^1 = \frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\mathbf{d}_2 + \frac{k^2\lambda_{13}}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\mathbf{e}_t, \tag{41d}$$

$$\mathbf{m}^2 = -\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\mathbf{d}_1 + \frac{k^2\lambda_{23}}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\mathbf{e}_t. \tag{41e}$$

Also, after a number of detailed but straightforward calculations that are outlined in Appendix A, it can be shown that the expressions (25), (28), (29) and (41) yield the results

$$\mathbf{n} = N\mathbf{e}_t + V\mathbf{e}_n + W\mathbf{e}_\tau + \left[\frac{(\mathbf{e}_t \cdot \lambda\mathbf{l}^\alpha)\mathbf{d}_\alpha}{d}\right], \tag{42a}$$

$$\mathbf{m} = T\mathbf{e}_t + M\mathbf{e}_\tau, \tag{42b}$$

where the forces $N$, $V$, $W$, and moments $T$, $M$ are defined by

$$N = \left[\frac{\partial\Sigma}{\partial d} - \frac{\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right], \quad V = -\left[\frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k}{d}\frac{\partial\Sigma}{\partial\beta}\right)\right], \tag{43a,b}$$

$$W = \left[\beta\frac{\partial\Sigma}{\partial\omega} - \frac{k\tau}{d}\frac{\partial\Sigma}{\partial\beta}\right], \tag{43c}$$

$$T = \left[k\frac{\partial\Sigma}{\partial\omega}\right], \quad M = \left[\frac{k}{d}\frac{\partial\Sigma}{\partial\beta}\right]. \tag{43d,e}$$

## 4. AN ALTERNATIVE FORMULATION

The constitutive equation (43a) for the axial force $N$ suggests that when the strain energy is a function of $\{d, \beta, \omega\}$ the effects on the kinetic quantities of extension and bending are naturally coupled. This is mainly due to the fact that $\kappa$ (or $\beta/k$) is the curvature of the stretched reference curve so it is naturally influenced by stretching of the rod. If the actual material response is such that the effects of extension and bending are uncoupled, it is more convenient to consider an alternative formulation based on the normalized curvature variable $\alpha$ defined by

$$\alpha = \beta d = k\kappa d. \tag{44}$$

It then follows that without loss in generality, the strain energy function $\Sigma$ in (39) can be written in the form

$$\Sigma(d, \beta, \omega, \xi) = \tilde{\Sigma}(d, \alpha, \omega, \xi). \tag{45}$$

Consequently,

$$\frac{\partial\Sigma}{\partial d} = \frac{\partial\tilde{\Sigma}}{\partial d} + \frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\alpha}, \quad \frac{\partial\Sigma}{\partial\beta} = d\frac{\partial\tilde{\Sigma}}{\partial\alpha}, \quad \frac{\partial\Sigma}{\partial\omega} = \frac{\partial\tilde{\Sigma}}{\partial\omega}, \tag{46a,b,c}$$

so that the constitutive equations (43) reduce to

$$N = \left[\frac{\partial\tilde{\Sigma}}{\partial d}\right], \quad V = -\frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[k\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right], \tag{47a,b}$$

$$W = \left[\frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\omega} - (k\tau)\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right], \tag{47c}$$

$$T = \left[k\frac{\partial\tilde{\Sigma}}{\partial\omega}\right], \quad M = \left[k\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right], \tag{47d,e}$$

with the force $\mathbf{n}$ and moment $\mathbf{m}$ still being expressed in the forms (42). Comparison, of (47a, b) with (43a, b) reveals the simplicity of this alternative formulation.

Next, with the help of (30) and Appendix A, the equations of equilibrium (17a) and (26) become

$$\left[\frac{\mathrm{d}N}{\mathrm{d}\xi}-\frac{V\alpha}{k}\right]\mathbf{e}_t+\left[\frac{\mathrm{d}V}{\mathrm{d}\xi}+\frac{N\alpha}{k}-W\tau d\right]\mathbf{e}_n+\left[\frac{\mathrm{d}W}{\mathrm{d}\xi}+V\tau d\right]\mathbf{e}_\tau+\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{(\mathbf{e}_t\cdot\lambda\mathbf{l}^\alpha)\mathbf{d}_\alpha}{d}\right]+\lambda\mathbf{f}=0, \tag{48a}$$

$$\frac{\mathrm{d}T}{\mathrm{d}\xi}=\mathbf{d}_1\cdot\lambda\mathbf{l}^2-\mathbf{d}_2\cdot\lambda\mathbf{l}^1. \tag{48b}$$

For a dynamical theory the total derivatives with respect to $\xi$ in (48) are replaced by partial derivatives and the assigned fields $\mathbf{f}$ and $\mathbf{l}^\alpha$ are replaced by the expression $\hat{\mathbf{f}}$ and $\hat{\mathbf{l}}^\alpha$ defined in (6c, e). Also, initial and boundary conditions must be supplied. In this regard, and with reference to an arbitrary section $[\xi_1, \xi_2]$ (with $\xi_1 < \xi_2$) of the rod, it is noted that $\mathbf{n}$ and $\mathbf{m}$ are the force and moment, respectively, applied to the boundary $\xi = \xi_2$ which has an outward normal $\mathbf{e}_t$. Similarly, $(-\mathbf{n})$ and $(-\mathbf{m})$ are the force and moment, respectively, applied to the boundary $\xi = \xi_1$ which has an outward normal $(-\mathbf{e}_t)$.

Next, if attention is confined to equilibrium problems in which the rod is only subjected to forces and moments applied to its ends then the assigned fields $\mathbf{f}$ and $\mathbf{l}^\alpha$ vanish

$$\mathbf{f}=0,\quad \mathbf{l}^\alpha=0, \tag{49a,b}$$

the force $\mathbf{n}$ and moment $\mathbf{m}$ reduce to

$$\mathbf{n}=N\mathbf{e}_t+V\mathbf{e}_n+W\mathbf{e}_\tau,\quad \mathbf{m}=T\mathbf{e}_t+M\mathbf{e}_\tau, \tag{50a,b}$$

and the equilibrium equations (48) become

$$\frac{\mathrm{d}N}{\mathrm{d}\xi}-\frac{V\alpha}{k}=0,\quad \frac{\mathrm{d}V}{\mathrm{d}\xi}+\frac{N\alpha}{k}-W\tau d=0, \tag{51a,b}$$

$$\frac{\mathrm{d}W}{\mathrm{d}\xi}+V\tau d=0,\quad \frac{\mathrm{d}T}{\mathrm{d}\xi}=0. \tag{51c,d}$$

In order to better interpret the difference between the twist $\omega$ and the geometric torsion $\tau$ it is convenient to note that the spatial derivative of the Serret–Frenet triad can be characterized by a rotation vector $\boldsymbol{\tau}$ defined such that

$$\boldsymbol{\tau}=\tau d\mathbf{e}_t+\kappa d\mathbf{e}_\tau, \tag{52a}$$

$$\frac{\mathrm{d}\mathbf{e}_t}{\mathrm{d}\xi}=\boldsymbol{\tau}\times\mathbf{e}_t,\quad \frac{\mathrm{d}\mathbf{e}_n}{\mathrm{d}\xi}=\boldsymbol{\tau}\times\mathbf{e}_n,\quad \frac{\mathrm{d}\mathbf{e}_\tau}{\mathrm{d}\xi}=\boldsymbol{\tau}\times\mathbf{e}_\tau. \tag{52b,c,d}$$

Moreover, it is possible to define an alternative orthonormal triad $\mathbf{b}_i$ such that $\mathbf{b}_3$ is the tangent vector $\mathbf{e}_t$ and the spatial derivative of the triad is characterized by the rotation vector $\boldsymbol{\omega}$ which is the part of $\boldsymbol{\tau}$ that does not include the effect of geometric torsion

$$\boldsymbol{\omega}=\kappa d\mathbf{e}_\tau,\quad \frac{\mathrm{d}\mathbf{b}_i}{\mathrm{d}\xi}=\boldsymbol{\omega}\times\mathbf{b}_i,\quad \mathbf{b}_3=\mathbf{e}_t. \tag{53a,b,c}$$

This triad has the property that once $\mathbf{b}_i$ are specified at one end of the rod (say $\xi = 0$) then $\mathbf{b}_i$ are uniquely determined by the eqn (53b) even when the curvature $\kappa$ vanishes at points on the curve and the binormal $\mathbf{e}_\tau$ is undefined. Also, it can be shown using eqns (2), (30a, b) and (32) that

$$\omega = \mathbf{e}_t \times \frac{d\mathbf{e}_t}{d\xi}, \quad \frac{d\mathbf{b}_\alpha}{d\xi} = -\left[\frac{d\mathbf{e}_t}{d\xi}\cdot\mathbf{b}_\alpha\right]\mathbf{e}_t, \quad \frac{d\mathbf{b}_3}{d\xi} = \frac{d\mathbf{e}_t}{d\xi}. \tag{54a,b,c}$$

Thus, with the help of these definitions it is possible to express the vectors $\mathbf{d}_\alpha$ in terms of $\mathbf{b}_\alpha$ by defining the angle $\theta$ such that

$$\mathbf{d}_1 = \cos\theta\mathbf{b}_1 + \sin\theta\mathbf{b}_2, \quad \mathbf{d}_2 = -\sin\theta\mathbf{b}_1 + \cos\theta\mathbf{b}_2. \tag{55a,b}$$

Then, it follows from (10b), (38) and (54b) that

$$\frac{d\mathbf{d}_1}{d\xi} = \frac{d\theta}{d\xi}\mathbf{d}_2 + \omega\times\mathbf{d}_1, \quad \frac{d\mathbf{d}_2}{d\xi} = -\frac{d\theta}{d\xi}\mathbf{d}_1 + \omega\times\mathbf{d}_2, \tag{56a,b}$$

and that the angle $\theta$ is related to the twist by the expression

$$\omega = k\frac{d\theta}{d\xi}. \tag{57}$$

## 5. INTRINSIC FORMS OF EXACT INTEGRALS

As mentioned in the introduction, it is known (Antman, 1972; Whitman and DeSilva, 1974; Antman and Jordan, 1975) that exact integrals of the equilibrium equations exist for the simple case of a rod that is loaded only by forces and moments on its ends. Consequently, it is expected that exact integrals of the intrinsic equilibrium equations also exist for this case. Specifically, in the following it will be shown that two exact integrals of the intrinsic equilibrium equations (51) exist for a general strain energy function of the form (45) which can characterize an inhomogeneous rod. Also, an additional energy-type integral exists when the rod is homogeneous.

To this end, it is obvious that the integral of equation (51d) indicates that the torsional moment $T$ is constant pointwise along the rod

$$T = \left[k\frac{\partial\tilde{\Sigma}}{\partial\omega}\right] = \text{constant}. \tag{58}$$

The second integral is obtained by substituting the constitutive equations (47) into the equilibrium equation (51c) to deduce that

$$\left[k\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right]\frac{d\tau}{d\xi} + 2\tau\frac{d}{d\xi}\left[k\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right] - \frac{d}{d\xi}\left[\frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\omega}\right] = 0, \tag{59a}$$

$$\frac{d}{d\xi}\left[k\left(\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right)^2\tau\right] - \left[\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right]\frac{d}{d\xi}\left[\frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\omega}\right] = 0, \tag{59b}$$

$$\frac{d}{d\xi}\left[k\left(\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right)^2\tau - \frac{\partial\tilde{\Sigma}}{\partial\alpha}\frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\omega}\right] + \left[\frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\omega}\right]\frac{d}{d\xi}\left[\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right] = 0. \tag{59c}$$

Now, substituting (47) into the equilibrium equation (51a) yields the additional result that

$$\frac{\alpha}{d}\frac{d}{d\xi}\left[\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right] = -\frac{d}{d\xi}\left[\frac{\partial\tilde{\Sigma}}{\partial d}\right], \tag{60}$$

so that with the help of (58), the equation (59c) integrates exactly to obtain

$$\left[\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right]^2(k\tau)-\left[\frac{\partial\tilde{\Sigma}}{\partial\omega}\right]\left[\frac{\partial\tilde{\Sigma}}{\partial d}+\frac{\alpha}{d}\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right]=C_1=\text{constant}, \tag{61}$$

where the constant $C_1$ is determined by boundary conditions.

Next, consider an energy-type quantity $\Phi$ defined by

$$\Phi=d\frac{\partial\tilde{\Sigma}}{\partial d}+\alpha\frac{\partial\tilde{\Sigma}}{\partial\alpha}+\omega\frac{\partial\tilde{\Sigma}}{\partial\omega}-\tilde{\Sigma}, \tag{62}$$

which is an implicit function of $\xi$. Using the constitutive equations (47) and differentiating (62) with respect to $\xi$ it can be shown that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=d\left[\frac{\mathrm{d}N}{\mathrm{d}\xi}-\frac{V\alpha}{k}\right]+\frac{\omega}{k}\frac{\mathrm{d}T}{\mathrm{d}\xi}-\frac{\partial\tilde{\Sigma}}{\partial\xi}, \tag{63}$$

which with the help of the equilibrium equations (51) reduces to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=-\frac{\partial\tilde{\Sigma}}{\partial\xi}. \tag{64}$$

Consequently, for rods which are homogeneous in the sense that the mass per unit length $\lambda$ and the strain energy function $\psi$ are explicitly independent of the coordinate $\xi$ it follows that

$$\lambda=\text{constant},\quad \frac{\partial\bar{\psi}}{\partial\xi}=0,\quad \frac{\partial\hat{\psi}}{\partial\xi}=0, \tag{65a,b,c}$$

$$\frac{\partial\tilde{\Sigma}}{\partial\xi}=0,\quad \Sigma=\tilde{\Sigma}(d,\alpha,\omega), \tag{65d,e}$$

and that (64) integrates to yield

$$\Phi=d\frac{\partial\tilde{\Sigma}}{\partial d}+\alpha\frac{\partial\tilde{\Sigma}}{\partial\alpha}+\omega\frac{\partial\tilde{\Sigma}}{\partial\omega}-\tilde{\Sigma}=\text{constant}. \tag{66}$$

Substituting (49a) and the result (A16) into the expression (B3) of Appendix B it can be seen that (66) is an energy-type integral similar to that developed by Ericksen (1970) in a more general context. Moreover, it is emphasized that in contrast with the first two integrals (58) and (61), which are valid for nonhomogeneous rods, the integral (66) is restricted to homogeneous rods in the sense of (65).

Now, again using the constitutive equations (47), the remaining equilibrium equation (51b) reduces to

$$k^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right)\right]+\hat{f}(\alpha)=0, \tag{67a}$$

$$\hat{f}(\alpha)=\left[\alpha\left\{-\frac{\partial\tilde{\Sigma}}{\partial d}+(k\tau)\frac{\partial\tilde{\Sigma}}{\partial\omega}\right\}-(k\tau)^2 d\frac{\partial\tilde{\Sigma}}{\partial\alpha}\right]. \tag{67b}$$

For simplicity, the remainder of this paper will confine attention to a homogeneous rod in equilibrium which is subjected only to forces and moments on its ends. Also, it will be assumed that in its reference configuration the rod is stress-free and that $\xi$ measures the arc-length of the rod in this stress-free state. In particular, for a rod of reference length $L$ the coordinate $\xi$ will vary in the range $(0\leqslant\xi\leqslant L)$.

Assuming that the strain energy function $\tilde{\Sigma}$ in (65e) is suitably restricted for the implicit function theorem to be applicable, the integrals (58) and (66) yield algebraic equations which can be solved implicitly to determine $d$ and $\omega$ as functions of $\alpha$

$$d = \hat{d}(\alpha), \quad \omega = \hat{\omega}(\alpha). \tag{68a,b}$$

Then, (68) can be substituted into the integral (61) to determine $\tau$ as a function of $\alpha$

$$\tau = \hat{\tau}(\alpha). \tag{69}$$

Next, it is convenient to define the additional functions

$$g = \hat{g}(\alpha) = \frac{\partial \tilde{\Sigma}}{\partial \alpha}, \quad G = \hat{G}(\alpha) = \frac{1}{d}\frac{\mathrm{d}\hat{g}}{\mathrm{d}\alpha}\hat{f}(\alpha), \tag{70a,b}$$

$$F = \hat{F}(\alpha) = \int_{\alpha_0}^{\alpha} \hat{G}(\sigma)\,\mathrm{d}\sigma, \tag{70c}$$

where the functions on the right-hand sides of (70a, b) are determined by substituting the functional forms (68) and (69) for $\{d, \omega, \tau\}$ into the expressions after the derivatives have been evaluated, and $\alpha_0$ is a constant. Then, (67) can be written in the integrated form

$$\frac{k^2}{2}\left[\frac{1}{d}\frac{\mathrm{d}\hat{g}(\alpha)}{\mathrm{d}\alpha}\right]^2\left[\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\right]^2 + \hat{F}(\alpha) = C_2 = \text{constant}, \tag{71}$$

where $C_2$ is another constant which can absorb the influence of $\alpha_0$ in (70c). Now, (71) is a first order differential equation to determine $\alpha$ as a function of the convected coordinate $\xi$. In order to integrate this equation it is necessary to specify the four constants of integration $\{T, \Phi, C_1, C_2\}$ as well as the value of $\alpha$ at a point on the rod.

Next, the fixed points $\alpha^*$ of (67) are the zeros of the function $\hat{f}(\alpha)$ and the character of each fixed point is determined by the local value of the derivative of $\hat{f}(\alpha)$ (see Nayfeh and Mook, 1979).

$$\hat{f}(\alpha^*) = 0 \quad \text{for each fixed point}, \tag{72a}$$

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}\alpha}(\alpha^*) > 0 \quad \text{for a center}, \tag{72b}$$

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}\alpha}(\alpha^*) < 0 \quad \text{for a saddle point}. \tag{72c}$$

Also, for each fixed point the constant $C_2$ in (71) is determined by the equation

$$C_2 = \hat{F}(\alpha^*), \tag{73}$$

so that $\alpha$ remains constant at each point along the rod.

Once the quantities $\{d, \alpha, \tau\}$ are determined as functions of $\xi$, the curvature $\kappa$ is also determined (44), and a differential equation for the position vector $\mathbf{r}(\xi)$ of the rod's reference curve can be obtained. To this end, equations (2) and (30) are used to derive the expressions

$$\frac{d^2\mathbf{r}}{d\xi^2} = \frac{1}{d}\frac{d(d)}{d\xi}\frac{d\mathbf{r}}{d\xi} + \kappa d^2\mathbf{e}_n, \tag{74a}$$

$$\frac{d^3\mathbf{r}}{d\xi^3} = \frac{1}{d}\frac{d(d)}{d\xi}\frac{d^2\mathbf{r}}{d\xi^2} + \frac{d}{d\xi}\left[\frac{1}{d}\frac{d(d)}{d\xi}\right]\frac{d\mathbf{r}}{d\xi} + \frac{d(\kappa d^2)}{d\xi}\mathbf{e}_n + \kappa d^2\left[-\kappa\frac{d\mathbf{r}}{d\xi} + \tau d\mathbf{e}_\tau\right]. \tag{74b}$$

However, since $\mathbf{e}_t \times \mathbf{e}_n = \mathbf{e}_\tau$ it follows from (74a) that

$$\frac{1}{d}\frac{d\mathbf{r}}{d\xi} \times \frac{d^2\mathbf{r}}{d\xi^2} = \kappa d^2\mathbf{e}_\tau, \tag{75}$$

so the $\mathbf{e}_\tau$ can be eliminated from (74b) to obtain

$$\frac{d^3\mathbf{r}}{d\xi^3} = \left[\frac{1}{d}\frac{d(d)}{d\xi} + \frac{1}{\kappa d^2}\frac{d(\kappa d^2)}{d\xi}\right]\frac{d^2\mathbf{r}}{d\xi^2} + \tau\left[\frac{d\mathbf{r}}{d\xi} \times \frac{d^2\mathbf{r}}{d\xi^2}\right]$$
$$+ \left[\frac{d}{d\xi}\left\{\frac{1}{d}\frac{d(d)}{d\xi}\right\} - \left\{\frac{1}{d}\frac{d(d)}{d\xi}\right\}\frac{1}{\kappa d^2}\frac{d(\kappa d^2)}{d\xi} - (\kappa d)^2\right]\frac{d\mathbf{r}}{d\xi}. \tag{76}$$

This equation can be integrated subjected to specification of the boundary conditions

$$\mathbf{r}(0), \quad \frac{d\mathbf{r}}{d\xi}(0), \quad \frac{d^2\mathbf{r}}{d\xi^2}(0). \tag{77a,b,c}$$

In this regard, it is noted that the intrinsic functions $\{d, \kappa, \tau\}$ determine the shape of the rod's reference curve and the conditions (77) merely determine the six rigid-body degrees of freedom that orient the rod in space. Specifically, using (2) and (30a, b) it can be shown that

$$\frac{d\mathbf{r}}{d\xi} = d\mathbf{e}_t, \quad \frac{d^2\mathbf{r}}{d\xi^2} = \frac{d(d)}{d\xi}\mathbf{e}_t + d^2\kappa\mathbf{e}_n. \tag{78a,b}$$

Thus, three translational degrees of freedom are specified by (77a), two rotational degrees of freedom are specified by $\mathbf{e}_t(0)$, and one additional rotational degree of freedom is specified by $\mathbf{e}_n(0)$.

## 6. UNIFORM SOLUTIONS FOR CIRCULAR HELICAL RODS

The class of helical solutions has been discussed by Love (1944), Antman (1974) and by Whitman and DeSilva (1974), and is presented here in terms of the intrinsic formulation of the previous section. In view of the implicit functions (68) and the expression (69) it follows that each of the fixed points (72) yields constant values for the stretch $d$, the curvature $\alpha$, the twist $\omega$, and the geometric torsion $\tau$. These results represent uniform solutions of the equilibrium equations that characterize a circular helix of radius $r$ and pitch angle $\gamma$. Specifically, with reference to the cylindrical polar base vectors $\{\mathbf{e}_r(\theta), \mathbf{e}_\theta(\theta), \mathbf{e}_3\}$ the position vector $\mathbf{r}(\xi)$ of points on this helix can be expressed in the form

$$\mathbf{r}(\xi) = r\mathbf{e}_r(\theta) + z\mathbf{e}_3, \tag{79a}$$

$$\theta = \left[\frac{d}{r}\cos\gamma\right]\xi, \quad z = [d\sin\gamma]\xi. \tag{79b,c}$$

Moreover, it can easily be shown that the curvature $\kappa$ and the geometric torsion $\tau$ associated with (79) satisfy the equations

$$\kappa = \frac{\cos^2\gamma}{r}, \quad \tau = \frac{\sin\gamma\cos\gamma}{r}, \tag{80a,b}$$

$$r = \frac{\kappa}{\kappa^2+\tau^2}, \quad \sin 2\gamma = \frac{2\kappa\tau}{\kappa^2+\tau^2}. \tag{80c,d}$$

Also, it follows from (47b) that for these solutions the shearing force $V$ vanishes.

Since the strain energy function $\Sigma$ is assumed to be explicitly independent of the coordinate $\xi$, the constitutive equations tacitly assume that the values of $d$, $\alpha$, and $\omega$ are constant pointwise in the stress-free reference configuration. However, these constitutive equations do not necessarily assume that the rod is straight in its stress-free state. This means that it is possible to use these equations to develop a simple model for analyzing the response of a circular helical spring which has a stress-free reference state with

$$d = 1, \quad r = R, \quad \gamma = \Gamma, \quad \omega = 0, \tag{81a,b,c,d}$$

$$\kappa = \kappa_0 = \frac{\cos^2\Gamma}{R}, \quad \tau = \frac{\sin\Gamma\cos\Gamma}{R}. \tag{81e,f}$$

Under these conditions the strain energy function must be restricted so that the quantities $\{N, W, TM\}$ vanish when (81) are satisfied.

## 7. THE INEXTENSIBLE CASE

For the inextensible case the additional constraint equations

$$d = 1, \quad \mathbf{d}_3 \cdot \mathbf{w}_3 = 0, \tag{82a,b}$$

are imposed which indicate that the constraint response $p^{33}$ in (21a) no longer vanishes. This causes the contact force $\mathbf{n}$ to be given by the form (42a), where the quantity $N$ is replaced by the constraint response $n$ which is no longer determined by a constitutive equation of the form (47a), and $V$, $W$, $T$ and $M$ are replaced by the functions $v$, $w$, $t$ and $m$ so that

$$\mathbf{n} = n\mathbf{e}_t + v\mathbf{e}_n + w\mathbf{e}_\tau + \left[\frac{(\mathbf{e}_t \cdot \lambda\mathbf{l}^\alpha)\mathbf{d}_\alpha}{d}\right], \tag{83a}$$

$$\mathbf{m} = t\mathbf{e}_t + m\mathbf{e}_\tau. \tag{83b}$$

Also, the strain energy for inhomogeneous rods becomes a function of $\alpha$, $\omega$ and $\xi$ only

$$\Sigma = \sigma(\alpha, \omega, \xi), \tag{84}$$

where in view of (44) and (82a) the normalized curvature becomes ($\alpha = k\kappa$). Then, the equilibrium equations take the forms (48) with $\{N, V, W, T\}$ replaced by $\{n, v, w, t\}$ and $d$ replaced by 1. Moreover, under the conditions (49), $\mathbf{n}$ and $\mathbf{m}$ become

$$\mathbf{n} = n\mathbf{e}_t + v\mathbf{e}_n + w\mathbf{e}_\tau, \quad \mathbf{m} = t\mathbf{e}_t + m\mathbf{e}_\tau, \tag{85a,b}$$

the equilibrium equations reduce to

$$\frac{dn}{d\xi}-\frac{v\alpha}{k}=0,\quad \frac{dv}{d\xi}+\frac{n\alpha}{k}-w\tau=0, \tag{86a,b}$$

$$\frac{dw}{d\xi}+v\tau=0,\quad \frac{dt}{d\xi}=0, \tag{86c,d}$$

and the constitutive equations (47) are given by

$$v=-\frac{d}{d\xi}\left[k\frac{\partial\sigma}{\partial\alpha}\right],\quad w=\left[\alpha\frac{\partial\sigma}{\partial\omega}-(k\tau)\frac{\partial\sigma}{\partial\alpha}\right], \tag{87a,b}$$

$$t=\left[k\frac{\partial\sigma}{\partial\omega}\right],\quad m=\left[k\frac{\partial\sigma}{\partial\alpha}\right]. \tag{87c,d}$$

Following the development in Section 5, it can be shown that two exact integrals similar to (58) and (61) exist of the forms

$$t=\left[k\frac{\partial\sigma}{\partial\omega}\right]=\text{constant}, \tag{88a}$$

$$\left(\frac{\partial\sigma}{\partial\alpha}\right)^2(k\tau)-\frac{\partial\sigma}{\partial\omega}\left[n+\alpha\frac{\partial\sigma}{\partial\alpha}\right]=c_1=\text{constant}, \tag{88b}$$

for the general case of an inhomogeneous rod. Moreover, for homogeneous rods an additional energy-type integral exists of the form

$$\phi=n+\alpha\frac{\partial\sigma}{\partial\alpha}+\omega\frac{\partial\sigma}{\partial\omega}-\sigma=\text{constant}. \tag{89}$$

Then, with the help of the constitutive equations (87), the equilibrium equation (86b) reduces to

$$k^2\frac{d^2}{d\xi^2}\left[\frac{\partial\sigma}{\partial\alpha}\right]+\alpha\left[-n+(k\tau)\frac{\partial\sigma}{\partial\omega}\right]-(k\tau)^2\frac{\partial\sigma}{\partial\alpha}=0. \tag{90}$$

## 8. SUMMARY AND DISCUSSION

An intrinsic form of a constrained Bernoulli-Euler type theory of rods has been developed which allows the nonlinear elastic strain energy to be a general function of the extension, curvature and twist of the rod. When the assigned director couple $\mathbf{l}^\alpha$ vanishes, these intrinsic equations yield simple forms for the constitutive equations (47) that allow easy physical interpretation of the force $\mathbf{n}$ and moment $\mathbf{m}$ applied to the rod's end [see (50)]. In particular, (50b) shows that the bending moment $M$ is applied in the direction of the binormal $\mathbf{e}_\tau$ to the rod's reference curve. This is consistent with the physical interpretation that the rod bends locally in the plane containing the tangent vector $\mathbf{e}_t$ and the normal vector $\mathbf{e}_n$ to the rod's reference curve.

For an extensible rod, it has been shown that two exact integrals [(58) and (61)] of the equilibrium equations (51) exist for a general inhomogeneous rod and that an additional energy-type integral (66) exists for homogeneous rods. Thus, for homogeneous rods and suitable restrictions on the strain energy function (65e), the equations (58), (61) and (66) yield implicit functions of the curvature $\alpha$ for the stretch $d$, the twist $\omega$, and the geometric torsion $\tau$. Then, the remaining equilibrium equation (72) becomes a second order differential equation for determining $\alpha$. These equations require the specification of five constants which are related to boundary conditions. For example, these constants are determined if the values of

$$\left\{d, \alpha, \omega, \tau, \frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\right\}, \tag{91}$$

are known at some point on the rod.

For the inextensible rod, the stretch $d$ is constrained to be unity and the axial force $n$ becomes an arbitrary function which is determined by a boundary condition instead of a constitutive equation. The analysis of the equilibrium equations and the existence of exact integrals is similar to the extensible case.

In the remainder of this section special constitutive assumptions for homogeneous rods are considered in order to quantitatively exhibit some nonlinear effects due to the inclusion of extensional deformation and a more general constitutive equation for the bending moment. To this end, it is assumed that strain energy function takes the separable form

$$\tilde{\Sigma} = EA[D(d) + B(\alpha) + \Omega(\omega)], \tag{92}$$

where $D$, $B$, and $\Omega$ are arbitrary functions of their arguments, $E$ is Young's modulus, and $A$ is the constant area of the rod's cross section. It then follows from (47d) that the torsional moment $T$ is a function of the twist $\omega$ only so that the integral (58) requires the twist to be constant pointwise along the rod (which is not true for the general form of $\tilde{\Sigma}$). Furthermore, using (47a, e) the axial force $N$ and bending moment $M$ are given by

$$N = EA\frac{\mathrm{d}D}{\mathrm{d}d}, \quad M = EAk\frac{\mathrm{d}B}{\mathrm{d}\alpha}. \tag{93a,b}$$

For the inextensible theory of Section 7, the term $D(d)$ is omitted from the strain energy function (92) and $\sigma$ in (84) reduces to

$$\sigma = EA[B(\alpha) + \Omega(\omega)]. \tag{94}$$

Then, using (87d) the moment $m$ becomes

$$m = EAk\frac{\mathrm{d}B}{\mathrm{d}\alpha} \quad \text{with } \alpha = k\kappa, \tag{95}$$

and (89) can be used to obtain an equation for the force $n$ of the form

$$n = \phi + \sigma - \alpha\frac{\partial\sigma}{\partial\alpha} - \omega\frac{\partial\sigma}{\partial\omega}, \tag{96}$$

where $\phi$ is a constant of integration. Thus, since $\omega$ and $\phi$ are constants, eqn (96) requires

$$n = n_0 + EA\left[B(\alpha) - \alpha\frac{\mathrm{d}B}{\mathrm{d}\alpha}\right] \quad \text{with } \alpha = k\kappa, \tag{97}$$

where $n_0$ is a constant of integration.

Now, consider the simpler case when $B(\alpha)$ is a quadratic function of the form

$$B(\alpha)=\tfrac{1}{2}\alpha^2. \tag{98}$$

Then, using (44), (82a), (87), (94) and (98) it can be shown that the equations of equilibrium (86) reduce to the forms

$$\frac{\mathrm{d}n}{\mathrm{d}\xi}+EAk^2\kappa\frac{\mathrm{d}\kappa}{\mathrm{d}\xi}=0,\quad -EAk^2\frac{\mathrm{d}^2\kappa}{\mathrm{d}\xi^2}+n\kappa-t\kappa\tau+EAk^2\tau^2\kappa=0, \tag{99a,b}$$

$$-EAk^2\tau\frac{\mathrm{d}\kappa}{\mathrm{d}\xi}+\frac{\mathrm{d}}{\mathrm{d}\xi}[t\kappa-EAk^2\tau\kappa]=0,\quad \frac{\mathrm{d}t}{\mathrm{d}\xi}=0, \tag{99c,d}$$

which can be seen to be identical to the equations developed by Lu and Perkins (1994). Consequently, this analysis shows that their equations are exact for this inextensible case even when $\Omega(\omega)$ remains an arbitrary function.

Next, consider the extensional case where $D(d)$ is given by

$$D(d)=\tfrac{1}{2}(d-1)^2, \tag{100}$$

but the functional forms $B(\alpha)$ and $\Omega(\omega)$ remain general. For this case the integral (66) yields

$$\Phi=EA\left[\frac{1}{2}(d^2-1)+\alpha\frac{\mathrm{d}B}{\mathrm{d}\alpha}-B(\alpha)+\omega\frac{\mathrm{d}\Omega}{\mathrm{d}\omega}-\Omega(\omega)\right]. \tag{101}$$

However, since $\omega$ is constant, (101) can be solved for the stretch $d$ to obtain

$$d=\left[C_3+2B(\alpha)-2\alpha\frac{\mathrm{d}B}{\mathrm{d}\alpha}\right]^{1/2}, \tag{102}$$

where $C_3$ is a constant of integration. Also, using (100) the constitutive equation (93a) reduces to

$$N=EA(d-1). \tag{103}$$

Furthermore, assuming that $B(0)$ vanishes and that $\mathrm{d}B/\mathrm{d}\alpha$ is bounded at $\alpha=0$ it follows from (97), (102) and (103) that the forces $n$ and $N$ will be equal at $\alpha=0$ if $C_3$ is given by

$$C_3=\left[1+\frac{n_0}{EA}\right]^2. \tag{104}$$

For simplicity, in the following analysis, the constant $n_0$ and the function $B(\alpha)$ are taken in the forms

$$n_0=0,\quad B(\alpha)=\tfrac{1}{2}[\alpha^2+b\alpha^3], \tag{105a,b}$$

where $b$ is a constant controlling nonlinear bending effects. Using these functions the normalized axial forces [$N/EA$ in (103), and $n/EA$ in (97)] and normalized bending moments [$M/EAk$ in (93b), and $m/EAk$ in (95)] associated with the extensible and inextensible cases,

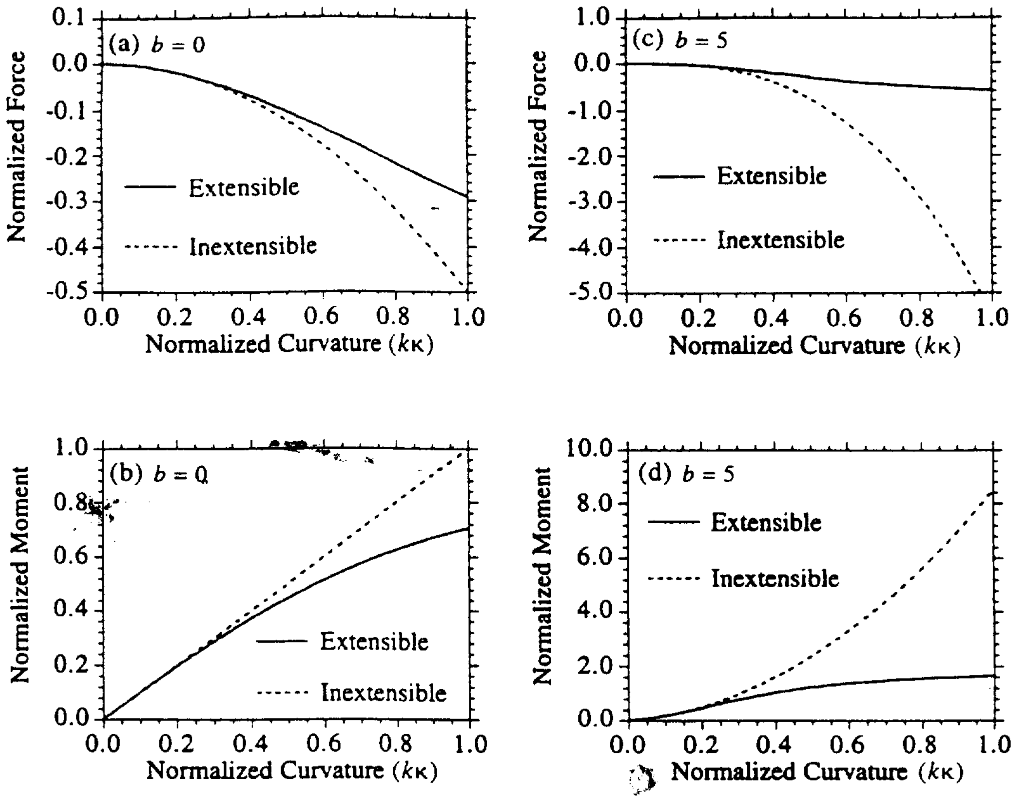


Fig. 1. Plots of the normalized axial force ($a, c$) and normalized bending moments ($b, d$) showing the effects of extensibility and nonlinearity in the function for bending energy ($b = 5$).

respectively, are plotted as functions of the normalized curvature $k\kappa$ in Fig. 1a and Fig. 1b for the linear function $B(\alpha)$ with $b = 0$ and in Fig. 1c and Fig. 1d for the nonlinear function $B(\alpha)$ with $b = 5$. The results in these figures indicate that extensibility and nonlinearity in the bending strain energy function become significant in the range of high curvature when a hockle would begin to close. Also, for these cases the effect of extension has a softening influence which causes the magnitudes of the axial force and bending moments to be smaller than those for the inextensible theory.

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## APPENDIX A: SOME DETAILS OF THE CALCULATIONS

In order to develop the results (42) and (43) it is necessary to determine expressions for the derivatives of $\mathbf{d}_\alpha$. To this end, it is noted that

$$\frac{\mathrm{d}\mathbf{d}_\alpha}{\mathrm{d}\xi} = \lambda_{\alpha j}\mathbf{d}^j, \tag{A1}$$

so that with the help of (22) and (34) it can be shown that

$$\frac{\mathrm{d}\mathbf{d}_1}{\mathrm{d}\xi} = \lambda_{12}\mathbf{d}_2 + \frac{\lambda_{13}}{d}\mathbf{e}_t, \tag{A2a}$$

$$\frac{\mathrm{d}\mathbf{d}_2}{\mathrm{d}\xi} = -\lambda_{12}\mathbf{d}_1 + \frac{\lambda_{23}}{d}\mathbf{e}_t. \tag{A2b}$$

Now, substitution of (A2) into (41) and use of (37), (38) and (40) yields

$$\hat{\mathbf{k}}^1 = -\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\left[-\lambda_{12}\mathbf{d}_1 + \frac{\lambda_{23}}{d}\mathbf{e}_t\right], \tag{A3a}$$

$$\hat{\mathbf{k}}^2 = \frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\left[\lambda_{12}\mathbf{d}_2 + \frac{\lambda_{13}}{d}\mathbf{e}_t\right], \tag{A3b}$$

$$\mathbf{k}^3 = \left[\frac{\partial\Sigma}{\partial d} - \frac{\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_t + \left[\frac{k\omega}{\beta d^4}\frac{\partial\Sigma}{\partial\beta}\right][-\lambda_{23}\mathbf{d}_1 + \lambda_{13}\mathbf{d}_2] + p^{3\alpha}\mathbf{d}_\alpha. \tag{A3c}$$

However, the expression (36) can be rewritten in the form

$$\lambda_{\alpha 3}\mathbf{d}_\alpha = -\frac{d^2\beta}{k}\mathbf{e}_n, \tag{A4}$$

which can be used to deduce that

$$-\lambda_{23}\mathbf{d}_1 + \lambda_{13}\mathbf{d}_2 = \mathbf{e}_t \times \left[-\frac{d^2\beta}{k}\mathbf{e}_n\right] = -\frac{d^2\beta}{k}\mathbf{e}_\tau. \tag{A5}$$

Thus, (A3c) reduces to

$$\mathbf{k}^3 = \left[\frac{\partial\Sigma}{\partial d} - \frac{\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_t - \left[\frac{\omega}{d^2}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_\tau + p^{3\alpha}\mathbf{d}_\alpha. \tag{A6}$$

Also, using (30b), (41) and (A2) it follows that

$$\frac{\mathrm{d}\mathbf{m}^1}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\right)\mathbf{d}_2 + \left(\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\right)\left[-\lambda_{12}\mathbf{d}_1 + \frac{\lambda_{23}}{d}\mathbf{e}_t\right] + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k^2\lambda_{13}}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right)\mathbf{e}_t + \left(\frac{k\lambda_{13}}{d^2}\frac{\partial\Sigma}{\partial\beta}\right)\mathbf{e}_n, \tag{A7a}$$

$$\frac{\mathrm{d}\mathbf{m}^2}{\mathrm{d}\xi} = -\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\right)\mathbf{d}_1 - \left(\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\right)\left[\lambda_{12}\mathbf{d}_2 + \frac{\lambda_{13}}{d}\mathbf{e}_t\right] + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k^2\lambda_{23}}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right)\mathbf{e}_t + \left(\frac{k\lambda_{23}}{d^2}\frac{\partial\Sigma}{\partial\beta}\right)\mathbf{e}_n. \tag{A7b}$$

Next, using (25), (34), (A3) and (A7) the constraint responses $p^{3\alpha}$ become

$$p^{31} = \frac{\mathbf{e}_t\cdot\lambda\mathbf{l}^1}{d} + \frac{k\lambda_{23}}{d^2}\frac{\partial\Sigma}{\partial\omega} + \frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k^2\lambda_{13}}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right), \tag{A8a}$$

$$p^{32} = \frac{\mathbf{e}_t\cdot\lambda\mathbf{l}^2}{d} - \frac{k\lambda_{13}}{d^2}\frac{\partial\Sigma}{\partial\omega} + \frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k^2\lambda_{23}}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right). \tag{A8b}$$

Thus, $\mathbf{k}^3$ can be written in the form

$$\mathbf{k}^3 = \left[\frac{\partial\Sigma}{\partial d} - \frac{\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_t - \left[\frac{\omega}{d^2}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_\tau + \left[\frac{\mathbf{e}_t\cdot\lambda\mathbf{l}^\alpha}{d}\right]\mathbf{d}_\alpha + \left[\frac{k}{d^2}\frac{\partial\Sigma}{\partial\omega}\right][\lambda_{23}\mathbf{d}_1 - \lambda_{13}\mathbf{d}_2] + \left[\frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{k^2}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right)\right][\lambda_{\alpha 3}\mathbf{d}_\alpha] + \left[\frac{k^2}{\beta d^4}\frac{\partial\Sigma}{\partial\beta}\right]\left[\frac{\mathrm{d}\lambda_{\alpha 3}}{\mathrm{d}\xi}\mathbf{d}_\alpha\right]. \tag{A9}$$

But, (22d), (30), (37), (38), (40), and (A2) can be used to write the derivative of (A4) as

$$\frac{d\lambda_{\alpha 3}}{d\xi}\mathbf{d}_\alpha = -\lambda_{13}\left[\lambda_{12}\mathbf{d}_2+\frac{\lambda_{13}}{d}\mathbf{e}_t\right]-\lambda_{23}\left[-\lambda_{12}\mathbf{d}_1+\frac{\lambda_{23}}{d}\mathbf{e}_t\right]-\frac{d}{d\xi}\left[\frac{d^2\beta}{k}\right]\mathbf{e}_n-\left[\frac{d^2\beta}{k^2}\right][-\beta d\mathbf{e}_t+k\tau d\mathbf{e}_\tau], \quad \text{(A10a)}$$

$$\frac{d\lambda_{\alpha 3}}{d\xi}\mathbf{d}_\alpha = -\frac{d}{d\xi}\left[\frac{d^2\beta}{k}\right]\mathbf{e}_n+\left[\frac{d^2\beta}{k^2}\right][\omega-k\tau d]\mathbf{e}_\tau, \quad \text{(A10b)}$$

so (A4), (A5) and (A10) can be substituted into (A9) to deduce that

$$\mathbf{k}^3 = \left[\frac{\partial\Sigma}{\partial d}-\frac{\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_t-\left[\frac{\omega}{d^2}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_\tau+\left[\frac{\mathbf{e}_t\cdot\lambda\mathbf{l}^\alpha}{d}\right]\mathbf{d}_\alpha+\left[\frac{k}{d^2}\frac{\partial\Sigma}{\partial\omega}\right]\left[\frac{d^2\beta}{k}\right]\mathbf{e}_\tau-\left[\frac{1}{d}\frac{d}{d\xi}\left(\frac{k^2}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right)\right]\left[\frac{d^2\beta}{k}\right]\mathbf{e}_n$$
$$+\left[\frac{k^2}{\beta d^4}\frac{\partial\Sigma}{\partial\beta}\right]\left[-\frac{d}{d\xi}\left\{\frac{d^2\beta}{k}\right\}\mathbf{e}_n+\left\{\frac{d^2\beta}{k^2}\right\}\{\omega-k\tau d\}\mathbf{e}_\tau\right], \quad \text{(A11a)}$$

$$\mathbf{k}^3 = \left[\frac{\partial\Sigma}{\partial d}-\frac{\beta}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_t-\left[\frac{1}{d}\frac{d}{d\xi}\left(\frac{k}{d}\frac{\partial\Sigma}{\partial\beta}\right)\right]\mathbf{e}_n+\left[\beta\frac{\partial\Sigma}{\partial\omega}-\frac{k\tau}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_\tau+\left[\frac{\mathbf{e}_t\cdot\lambda\mathbf{l}^\alpha}{d}\right]\mathbf{d}_\alpha. \quad \text{(A11b)}$$

The expression (A11b) then leads to the results (42a) and (43a–c).

To derive the expression (42b) it is necessary to substitute (41d, e) into (28a) and to use (A5) to obtain

$$\mathbf{m} = \left[k\frac{\partial\Sigma}{\partial\omega}\right]\mathbf{e}_t+\left[\frac{k^2}{\beta d^3}\frac{\partial\Sigma}{\partial\beta}\right](\lambda_{23}\mathbf{d}_1-\lambda_{13}\mathbf{d}_2), \quad \text{(A12a)}$$

$$\mathbf{m} = \left[k\frac{\partial\Sigma}{\partial\omega}\right]\mathbf{e}_t+\left[\frac{k}{d}\frac{\partial\Sigma}{\partial\beta}\right]\mathbf{e}_\tau. \quad \text{(A12b)}$$

Also, with the help of (34), (A3), (A4) and (A7) it follows that the director momentum eqn (26b) reduces to

$$\mathbf{d}_1\cdot\left[\lambda\mathbf{l}^2+\frac{d\mathbf{m}^2}{d\xi}\right] = \mathbf{d}_2\cdot\left[\lambda\mathbf{l}^1+\frac{d\mathbf{m}^1}{d\xi}\right], \quad \text{(A13a)}$$

$$\mathbf{d}_1\cdot\lambda\mathbf{l}^2-\frac{d}{d\xi}\left[\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\right]+\left[\frac{k\lambda_{23}}{d^2}\frac{\partial\Sigma}{\partial\beta}\right](\mathbf{d}_1\cdot\mathbf{e}_n) = \mathbf{d}_2\cdot\lambda\mathbf{l}^1+\frac{d}{d\xi}\left[\frac{k}{2}\frac{\partial\Sigma}{\partial\omega}\right]+\left[\frac{k\lambda_{13}}{d^2}\frac{\partial\Sigma}{\partial\beta}\right](\mathbf{d}_2\cdot\mathbf{e}_n). \quad \text{(A13b)}$$

But, (A5) and the fact that $\mathbf{e}_n$ and $\mathbf{e}_\tau$ are orthogonal can be used to rewrite (A13b) in the form

$$\frac{d}{d\xi}\left[k\frac{\partial\Sigma}{\partial\omega}\right] = \mathbf{d}_1\cdot\lambda\mathbf{l}^2-\mathbf{d}_2\cdot\lambda\mathbf{l}^1, \quad \text{(A14)}$$

which yields the equilibrium equation (48b).

Finally, equations (7), (22d), (30a), (38), (41d, e), (A2) and (A11b) are used to obtain

$$\mathbf{k}^3\cdot\mathbf{d}_3+\mathbf{m}^\alpha\cdot\frac{d\mathbf{d}_\alpha}{d\xi} = \left[d\frac{\partial\Sigma}{\partial d}-\beta\frac{\partial\Sigma}{\partial\beta}\right]+\left[k\lambda_{12}\frac{\partial\Sigma}{\partial\omega}\right]+\left[\frac{k^2\lambda_{\alpha3}\lambda_{\alpha3}}{\beta d^4}\frac{\partial\Sigma}{\partial\beta}\right], \quad \text{(A15a)}$$

$$\mathbf{k}^3\cdot\mathbf{d}_3+\mathbf{m}^\alpha\cdot\frac{d\mathbf{d}_\alpha}{d\xi} = \left[d\frac{\partial\Sigma}{\partial d}+\omega\frac{\partial\Sigma}{\partial\omega}\right]. \quad \text{(A15b)}$$

Next, using (46) it can be shown that

$$\mathbf{k}^3\cdot\mathbf{d}_3+\mathbf{m}^\alpha\cdot\frac{d\mathbf{d}_\alpha}{d\xi} = \left[d\frac{\partial\tilde{\Sigma}}{\partial d}+\alpha\frac{\partial\tilde{\Sigma}}{\partial\alpha}+\omega\frac{\partial\Sigma}{\partial\omega}\right], \quad \text{(A16)}$$

which is used to derive a result related to the expression (66).

## APPENDIX B: ERICKSEN'S ENERGY INTEGRAL

Ericksen (1970) has considered a general class of Cosserat rod theories and has developed an energy integral for homogeneous rods in equilibrium. For convenience of the reader this energy integral will be rederived in terms of the variables used in this paper. First, it is recalled that an essential condition required for the energy integral to exist is that the rod be homogeneous in the sense of (65). Now, with the help of this condition and the results (15) it can be shown that

$$\frac{d(\lambda\psi)}{d\xi} = \lambda\frac{\partial\bar{\psi}}{\partial \mathbf{d}_i}\cdot\frac{d\mathbf{d}_i}{d\xi} + \lambda\frac{\partial\bar{\psi}}{\partial(d\mathbf{d}_\alpha/d\xi)}\cdot\frac{d}{d\xi}\left[\frac{d\mathbf{d}_\alpha}{d\xi}\right]$$

$$= \hat{\mathbf{k}}^i\cdot\frac{d\mathbf{d}_i}{d\xi} + \hat{\mathbf{m}}^\alpha\cdot\frac{d}{d\xi}\left[\frac{d\mathbf{d}_\alpha}{d\xi}\right]. \quad \text{(B1)}$$

Moreover, since the constraint responses $\bar{\mathbf{k}}^i$ and $\bar{\mathbf{m}}^\alpha$ are workless, it follows from (13) that the expression (B1) reduces

$$\frac{d(\lambda\psi)}{d\xi} = \mathbf{k}^i\cdot\frac{d\mathbf{d}_i}{d\xi} + \mathbf{m}^\alpha\cdot\frac{d}{d\xi}\left[\frac{d\mathbf{d}_\alpha}{d\xi}\right]. \quad \text{(B2)}$$

Next, to determine the energy-type integral it is further assumed that the assigned field $\mathbf{f}$ is constant and the assigned fields $\mathbf{l}^\alpha$ vanish. Then, the functional

$$\Phi = \lambda\mathbf{f}\cdot[\mathbf{r}-\mathbf{r}(0)] + \mathbf{k}^3\cdot\mathbf{d}_3 + \mathbf{m}^\alpha\cdot\frac{d\mathbf{d}_\alpha}{d\xi} - \lambda\psi, \quad \text{(B3)}$$

is considered where $\mathbf{r}(0)$ represents the location of an end of the rod. Differentiation of (B3) with respect to $\xi$ and use of (B2) yields the expression

$$\frac{d\Phi}{d\xi} = \left[\lambda\mathbf{f} + \frac{d\mathbf{k}^3}{d\xi}\right]\cdot\mathbf{d}_3 + \left[-\mathbf{k}^\alpha + \frac{d\mathbf{m}^\alpha}{d\xi}\right]\cdot\frac{d\mathbf{d}_\alpha}{d\xi} = 0, \quad \text{(B4)}$$

which is seen to vanish for equilibrium (17) with vanishing $\mathbf{l}^\alpha$.

This proves that the quantity $\Phi$ in (B3) is an energy-type integral that is constant at each point on the rod. In particular, it is emphasized that this result is valid for general nonlinear elastic constitutive equations that include the effects of tangential extension, bending, normal cross-sectional extension, tangential shear deformation, and normal cross-sectional shear deformation (Naghdi and Rubin, 1984). However, as previously mentioned, it is essential that the rod be homogeneous in the sense of (65).

Further, with regard to superposed rigid body deformation it is assumed that $\lambda$ and $\psi$ are unaffected by superposed rigid body deformation and that $\mathbf{r}$, $\mathbf{f}$, $\mathbf{k}^i$ and $\mathbf{m}^\alpha$ transform to their superposed values $\mathbf{f}^+$, $\psi^+$, $\mathbf{k}^{i+}$, $\mathbf{m}^{\alpha+}$ by the transformation relations

$$\mathbf{r}^+ = \mathbf{c}+\mathbf{Q}\mathbf{r}, \quad \mathbf{f}^+ = \mathbf{Q}\mathbf{f}, \quad \mathbf{k}^{i+} = \mathbf{Q}\mathbf{k}^i, \quad \mathbf{m}^{\alpha+} = \mathbf{Q}\mathbf{m}^\alpha, \quad \text{(B5a,b,c,d)}$$

where $\mathbf{Q}$ is a constant proper orthogonal tensor and $\mathbf{c}$ is a constant vector. Then, the value of $\Phi$ transforms to $\Phi^+$ such that

$$\Phi^+ = \Phi. \quad \text{(B6)}$$

Also, it is noted that if $\mathbf{l}^\alpha$ are constants, then a term like $(\lambda\mathbf{l}^\alpha\cdot d\mathbf{d}_\alpha/d\xi)$ can be added to the expression $\Phi$ in (B3) without changing the result that $\Phi$ is constant pointwise. However, this was not done because constant fields $\mathbf{l}^\alpha$ do not appear to be physically realizable since $\mathbf{l}^\alpha$ are associated with the directions defined by the directors $\mathbf{d}_\alpha$ and these directions do not necessarily remain constant.